

# A Renormalization Group Study of Interacting Helical Liquid : Physics of Majorana Fermion

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## Abstract

The physics of Majorana fermion occurs at the edge and interface of many one-dimensional quantum many body system with gapped excitation spectrum. We present the renormalization group equations for strongly interacting helical liquid. We present the results of both Majorana-Ising topological quantum phase transition and also Berezinskii-Kosterlitz-Thouless topological phase transition. We show that the Majorana-Ising topological quantum phase transition for non-interacting case is an exactly solvable line but in presence of interaction the system has no exactly solvable line. We study the effect of umklapp scattering on the renormalization group flow diagrams and also the condition for the appearance of Majorana fermion modes.

Keywords: Topological Insulator, Helical Liquid, Topological Quantum Phase Transition

## I. INTRODUCTION

The physics of Majorana fermion is the subject of intense research interest in quantum many body system near a deca [1-31]. Majorana fermion appears in the topologically ordered state which is one of the important focus of research in condensed matter physics. These states are characterized by the topological invariant quantities, Chern number and Winding number. Kitaev [3] and the other research group have proposed the physical realization of Majorana fermion as a edge state of the one dimensional system that include one-dimensional superconductors, semiconductor quantum wire, proximity coupled to superconductor, the cold atom trapped in one-dimension. The physics and the control technology of Majorana fermion are the most important properties of non-abelian quantum computation, the quantum Hall states are the example of topological states [11]. Therefore the physics of Majorana fermion is almost ubiquitous in different quantum many body system [1-31].

In this research paper, we study for the existence of Majorana fermion mode in interacting helical liquid by using the renormalization group theory method.

Before we proceed further we would like to state the basic properties of Helical liquid very briefly [9-12,16-31]. The physics of helical liquid is interesting for the following properties. It is generally originated from the quantum spin Hall effect in a system with or without Landau levels. In the quantum spin Hall effect the left movers in the edge are connected with the down spin and the right movers with the up spin and the transport process is quantized. This physics is generally termed as a "Helical liquid" which describes the connection between the spin and momentum. It does not break time reversal invariance which occurs in chiral Luttinger liquid. The helical liquid has even number of time reversal pair whereas the spinful Luttinger liquid has odd number of pairs. The physics of helical liquid occurs in the surface state of a two dimensional insulator a proximity coupled semiconductor quantum wire subject to the spin orbit interaction.

Here we describe the basic mathematical analysis of Helical liquid. It is well known to us that the low energy excitation in the one dimensional quantum many body system occurs near to the adjacent region of Fermi points. Therefore one can write the fermionic field

operator as [33]

$$\psi_\sigma(x) = \frac{1}{\sqrt{L}} \left[ \sum_{-\Lambda < k - k_F < \Lambda} e^{ik \cdot x} \psi_\sigma + \sum_{-\Lambda < k + k_F < \Lambda} e^{ik \cdot x} \psi_\sigma \right]. \quad (1)$$

Where  $\Lambda$  is the cut-off around the Fermi momentum ( $k_F$ ). We may consider the first term as a right mover ( $k > 0$ ) and the second term as a left mover ( $k < 0$ ). One can write the fermionic field with spin  $\sigma$  as  $\psi_\sigma(x) = \psi_{R\sigma}(x) + \psi_{L\sigma}(x)$ . For the low energy elementary excitations one can write the Hamiltonian as

$$H_0 = \int \frac{dk}{2\pi} v_F [\psi_{R\uparrow}^\dagger (i\partial_x) \psi_{R\uparrow} - i\psi_{L\downarrow}^\dagger (i\partial_x) \psi_{L\downarrow} + \psi_{R\downarrow}^\dagger (i\partial_x) \psi_{R\downarrow} - i\psi_{L\uparrow}^\dagger (i\partial_x) \psi_{L\uparrow}]. \quad (2)$$

The first term within the first bracket is one of the Kramer's pair and the second term within the first bracket is the another Kramer's pair. Therefore one of the Kramer's pair in the upper edge and the other one at the lower edge of the system. The total fermionic field,  $\psi(x) = \psi_{R\uparrow} + \psi_{L\downarrow}$ , which can be expressed as a spinor  $\psi = (\psi_{L\downarrow} \ \psi_{R\uparrow})^T$ . This is the simple mathematical picture of a helical liquid, where the spin is determined by the direction of the particle. The non-interacting part of the helical liquid for single edge in terms of spinor field is

$$H_0 = \psi^\dagger (iv_F \partial_x \sigma^z - \mu) \psi. \quad (3)$$

The authors of Ref.[12] have introduced a model Hamiltonian to study the Majorana fermion in strongly interacting helical liquids. Here we describe the model very briefly of Ref. [12]. The authors have described a low-dimensional quantum many body system of topological insulator in the proximity of s-wave superconductor and an external magnetic field along the edge of this system. The additional terms in the Hamiltonian is

$$\delta H = \Delta \psi_{L\downarrow} \psi_{R\uparrow} + B \psi_{L\downarrow}^\dagger \psi_{L\uparrow} + h.c. \quad (4)$$

Where  $\Delta$  is the proximity induce superconducting gap and  $B$  is the applied magnetic field along the edge of the sample. The Hamiltonian  $H_0$  is the time reversal invariant but the Hamiltonian  $\delta H$  is not time reversal invariant.

Now we consider the generic interaction which are time reversal, the authors of Ref. [12], have considered the two particle forward and umklapp scattering as

$$H_{fw} = g_2 \psi_{L\downarrow}^\dagger \psi_{L\downarrow} \psi_{R\uparrow}^\dagger \psi_{R\uparrow}. \quad (5)$$

In the umklapp scattering term, we write the umklapp term for half filling following the Wu, Bernevig and Zhang [25], the umklapp term at the half filling in a point splitted form. The point splitted version can be described as a regularization of the theory. Therefore the umklapp term become

$$H_{um} = -g_u \int dx \psi_{R\uparrow}^\dagger(x) \psi_{R\uparrow}^\dagger(x+a) \psi_{L\downarrow}^\dagger(x) \psi_{L\downarrow}^\dagger(x+a). \quad (6)$$

Where  $a$  is the lattice constant. This analytical expression gives a regularized theory using the lattice constant  $a$  as a ultraviolet cut-off.

If we use the first order Taylor series expansion of the fermionic field

$$\psi_{R\uparrow}^\dagger(x+a) \sim \psi_{R\uparrow}^\dagger(x) + a \partial_x \psi_{R\uparrow}^\dagger(x). \quad (7)$$

If we use this expansion for the umklapp scattering term then we produce the analytical expression for umklapp in a conventional form of the authors of Ref.[25].

$$H_{um} = g_u \psi_{L\downarrow}^\dagger \partial_x \psi_{L\downarrow}^\dagger \psi_{R\uparrow} \partial_x \psi_{R\uparrow} + h.c \quad (8)$$

Therefore the total Hamiltonian of the system is

$$H = H_0 + H_{fw} + H_{um} + \delta H. \quad (9)$$

Now we can write the above Hamiltonian as,  $H_{XYZ} = \sum_i H_i$  (up to a constant) [12].

$$H_i = \sum_{\alpha} J_{\alpha} S_i^{\alpha} S_{i+1}^{\alpha} - [\mu + B(-1)^i] S_i^z. \quad (10)$$

Where  $J_{x,y} = J \pm \Delta > 0$  and  $J = v_F$  and  $J_z > 0$ .

Here we write down the bosonized form of this model Hamiltonian, the detail derivation is relegated to appendix.

$$H = \frac{v}{2} \left( \frac{1}{K} ((\partial_x \phi)^2 + K (\partial_x \theta)^2) - \left( \frac{\mu}{\pi} \right) \partial_x \phi \right. \\ \left. + \frac{B}{\pi} \cos(\sqrt{4\pi} \phi) - \frac{\Delta}{\pi} \cos(\sqrt{4\pi} \theta) + \frac{g_u}{2\pi^2} \cos(4\sqrt{\pi} \phi) \right) \quad (11)$$

There is a sign mismatch between the present study and sign of Ref. [12], please see the appendix for detail explanation.

The author of Ref. [12] have studied the quantum phase diagram of this model Hamiltonian

based on the Abelian bosonization method. But here we study the same Hamiltonian by using the RG method for the following reasons. It is very clear from the continuum field theoretical study that our model Hamiltonian contains two strongly relevant and mutually nonlocal perturbation over the Gaussian (critical) theory. In such a situation the strong coupling fixed point is usually determined by the most relevant perturbation whose amplitude grows up according to its Gaussian scaling dimensions and it is not much affected by the less relevant coupling terms. However, this is not the general rule if the two operators exclude each other, i.e., if the field configurations which minimize one perturbation term do not minimize the other. In this case interplay between the two competing relevant operators can produce a novel quantum phase transition through a critical point or a critical line. Therefore, we would like to study the RG equation to interpret the quantum phases of the system.

## ANALYSIS OF RENORMALIZATION GROUP EQUATIONS

We now study how the parameters  $B$ ,  $\Delta$  and  $K$  flow under RG. through the operator product expansion. So the RG equations for their coefficients therefore are coupled to each other. We use operator product expansion to derive these RG equations which is independent of boundary condition [32]. In our derivation, we consider two operators,  $X_1 = e^{(ia_1\phi+ib_1\theta)}$  and  $X_2 = e^{(ia_2\phi+ib_2\theta)}$ . In the RG procedure, one can write these two field operators as a sum of fast and slow mode fields. In the fast field, the momentum range is  $\Lambda e^{-dl} < K < \Lambda$  and for the slow field  $K < \Lambda e^{-dl}$ , where  $\Lambda$  is the momentum cut-off,  $dl$  is the change in the logarithmic scale. The next step is the integration of the fast field for the operators  $X_1$  and  $X_2$ , it yields a third operator at the same space time point,  $X_3 = e^{i(a_1+a_2)\phi+i(b_1+b_2)\theta}$ . The prefactor of  $X_3$  can be found by the relation,  $X_1 X_2 \sim e^{-(a_1 a_2 + b_1 b_2) \frac{dl}{2\pi}} X_3$ . Our Hamiltonian consists of two operators, if we consider  $l_1$  and  $l_2$  as the coefficient of the operators  $X_1$  and  $X_2$  respectively. Then the RG expressions for  $\frac{dX_3}{dl}$  contains the term  $(a_1 a_2 + b_1 b_2) \frac{l_1 l_2}{2\pi}$ . This is the procedure to derive these RG equations.

In the RG process, one can write RG equations themselves are established in a perturbative expansion in coupling constant ( $g(l)$ ). They cease to be valid beyond a certain length scale,

where  $g(l) \sim 1$  [33].

$$\begin{aligned}
\frac{dB}{dl} &= (2 - K)B + 4Kg_u B, \\
\frac{d\Delta}{dl} &= (2 - \frac{1}{K})a, \\
\frac{dg_u}{dl} &= (2 - 4K)g_u + 2KB^2 \\
\frac{dK}{dl} &= \frac{a^2}{4} - K^2\Delta^2.
\end{aligned} \tag{12}$$

Before we proceed further to study the detail analysis of these four RG equations. Here at first we consider the absence of umklapp scattering, therefore the four RG equations reduced to three RG equations. The RG equations for the coefficients of Hamiltonian  $H_{XYZ}$  are

$$\begin{aligned}
\frac{d\Delta}{dl} &= (2 - \frac{1}{K})\Delta, \\
\frac{dB}{dl} &= (2 - K)B \\
\frac{dK}{dl} &= \frac{\Delta^2}{4} - K^2B^2.
\end{aligned} \tag{13}$$

Here our main intention is to study these three equations explicitly to explain the different quantum phases of the system and also the Majorana-Ising quantum phase transition. After few steps of calculation, one can also write the above three RG equations as

$$\begin{aligned}
\frac{d\Delta}{dl} &= (2 - \frac{1}{K})\Delta, \\
\frac{dB}{dl} &= (2 - K)B \\
\frac{d\ln K}{dl} &= K^{-1}\Delta^2 - KB^2.
\end{aligned} \tag{14}$$

where  $l = \ln[\frac{\Lambda}{\Lambda_0}]$  is the flow parameter,  $\Lambda_0$  is the initial value of the parameter. It is very clear from the above RG equations which reflect the duality in our helical liquid model system. The duality is the following.  $\phi \leftrightarrow \theta$ ,  $K \leftrightarrow K^{-1}$  and  $\Delta \leftrightarrow B$ . This duality in RG equations is actually from the duality, i.e., if we interchange the  $B$  and  $\Delta$  and simultaneously the field  $\phi$  and  $\theta$  then the Hamiltonian remains invariant of the bosonized Hamiltonian.

These RG equations have trivial ( $\Delta^* = 0 = B^*$ ) fixed points for any arbitrary  $K$ . Apart from that these RG equations have also two non-trivial fixed lines,  $\Delta = B$  and  $\Delta = -B$  for

$K = 1$ .

It is very clear from the above RG equations, in the vicinity of  $K = 1$ , one can find the non-trivial critical points, where both of the coupling term are become the relevant coupling, i.e., the coupling constant flowings off to the strong coupling limit, where each of them acquires a mass term.

The coupling strengths  $\Delta$  and  $B$  are related to the dual fields,  $\theta$  and  $\phi$  that drives the system to the different ground state, one is in the proximity induced superconducting gap and the other is in the applied magnetic field induce ferromagnetic phase [12, 25]. The interplay between them give rise to the second order quantum phase transition at the intermediate couplings. We have already discussed that the  $K = 1$ , the self dual point and this point is also the only exactly solvable point.

In our RG calculation, we also predict the Majorana-Ising transition. Before we proceed further we would like to explain the basic aspect of Majorana-Ising quantum phase transition.

$$\delta H = B\psi_{L\downarrow}^\dagger\psi_{R\uparrow} + \Delta\psi_{L\downarrow}\psi_{R,\uparrow} + h.c. \quad (15)$$

We recast the fermionic field interms of the Majorana fields,  $\psi_{L\downarrow}(x) = \frac{1}{2}(i\chi_1(x) + \chi_2(x))$ .  $\psi_{R\uparrow}(x) = \frac{1}{2}(\tilde{\chi}_1(x) + i\tilde{\chi}_2(x))$ . The total Hamiltonian,  $H = H_0 + \delta H$ , become

$$H = \sum_{i=1,2} (i\chi_i \frac{v_F}{2} \partial_x \chi_i - i\tilde{\chi}_i \frac{v_F}{2} \partial_x \tilde{\chi}_i + im_i \chi_i \tilde{\chi}_i). \quad (16)$$

Where  $m_{1,2} = \Delta \mp B$  (here  $\Delta > 0$ ). At  $\Delta = B$ , one of the two Majorana fermion modes become gapless which is the signature of bulk Majorana-Ising quantum phase transition.

We will explore this transition explicitly during the renormalization group study.

Now we study the scaling theory for this problem. It is well known to all of us that the critical theory is invariant under the rescaling, the singular part of the free energy density satisfies the following scaling relations.

$$f_s[\Delta, B] = e^{-2l} f_s[e^{(2-1/K)l} \Delta, e^{(2-K)l} B]. \quad (17)$$

The scale  $l$  can be fixed from the following analytical relation,  $e^{(2-1/K)l^*} \Delta = 1$ . After a few substitution, we arrive at the follwing relation.

$$f_s[\Delta, B] = \Delta^{2/(2-1/K)} f_s[1, \Delta^{-(2-K)/(2-1/K)} B]. \quad (18)$$

The phase transition of the system occurs when the coupling strength satisfy the following relation:

$$\Delta^{-(2-K)/(2-1/K)} B \sim 1 \quad (19)$$

The phase boundary between this two quantum derive by using the above relation. When  $K = 1$ , (non-interacting case), the above phase boundary relation is  $\Delta = B$ , which we have already discuss in the previous section, when we introduce the basic physics of Majorana-Ising topological quantum phase transition.

Now we discuss, the effect of repulsive ( $K < 1$ ) and attractive ( $K > 1$ ) region of this phase when the umklapp interaction is present. These results are depicted in the fig.1. It observes from our study that the phase boundary exists for the non-interating and also for the interacting one. The phase boundary is a exactly solvable line for the non-interacting case but for the interacting case it is not exactly solvable line.

One can understand this shift of this phase boundary from the following mathematical analysis. Generally one can consider the interaction between the Majorana fermions as  $H_{int} \sim \chi_1 \bar{\chi}_1 \chi_2 \bar{\chi}_2$ . In the mean field level, one can do the following approximation,  $\chi_2 \bar{\chi}_2 \rightarrow \langle \chi_2 \bar{\chi}_2 \rangle$ . Thus it is clear from the mean-field analysis that one can absorab the effect of interaction as a redefine mass in the system (Eq.16), which is shifting the phase boundary.

Here we discuss very briefly based on this toy model about the nature of quantum phases during the topological quantum phase transition.

In our toy model, we consider the simple case, i.e., the umklapp term is absent and at the point  $\Delta = J$ . The complete Hamiltonian reduce to transverse Ising model.

$$H_i = 2\Delta s_i^x s_{i+1}^x - B s_i^z \quad (20)$$

when  $B < \Delta$ , the discrete Ising symmetry is spontaneously broken which yields a doubly degenerate ordered phase which is proximity effect induce superconducting gap state which form the Majorana fermion mode excitations at the edge of the system. For  $B > \Delta$ , the magnetic field induce ferromagnetic state along the direction of magnetic field.



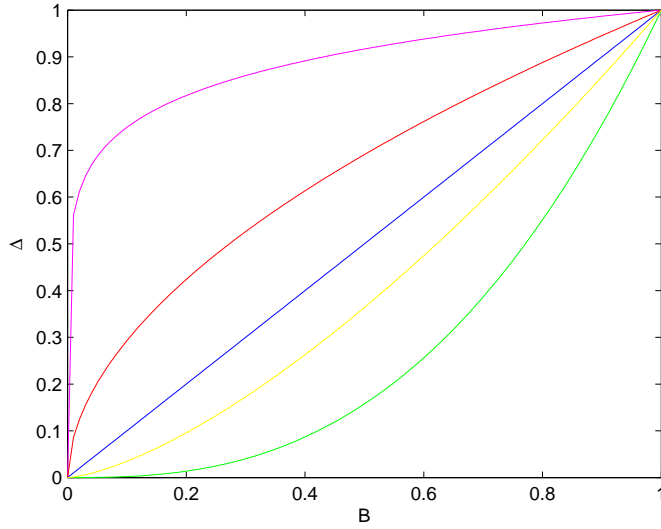


FIG. 1: (Colour online.) This figure presents the phase boundary between the two different phases  $\Delta$  and  $B$ . The color blue, magenta, red, green and yellow for  $K = 1, 0.55, 0.75, 1.5, 1.2$  respectively. The blue line is the exactly solvable line.

## RESULTS AND DISCUSSIONS BASED ON RG EQUATIONS STUDY

In fig.2, we present the RG flow diagram for  $\Delta$  with  $B$  for  $K = 1$ . It reveals from our study that both the couplings ( $\Delta$  and  $B$ ) are flowing off to the strong coupling phase. Here both the coupling terms are relevant, i.e., both of the coupling terms flowing off to the strong coupling phase.

In fig.3, we present the RG flow diagram for  $\Delta$  with  $B$  for  $K = 0.45$ . For this value of  $K$ , system is in the repulsive regime. According to the analysis of Abelian bosonization, the coupling term  $\Delta$  is irrelevant but the RG flow diagram of our study shows that the  $\Delta$  is non-zero but the magnitude has reduced. This is the advantage of the study of RG flow diagram over the Abelian bosonization study. It is also clear from this study that for this value of  $K$ , the magnitude of  $\Delta$  is almost the same it's initial value, i.e., the values of  $\Delta$ s are not changing during the RG flow. The proximity induce superconducting gap is non-zero, i.e, the Majorana fermion mode exists at the edge of topological insulator of the system.

In fig.4, we present the RG flow diagram  $\Delta$  with  $B$  for  $K = 0.2$ , i.e, the system is in the strongly repulsive regime compare to the  $K = 0.45$ . It reveals from our study that the

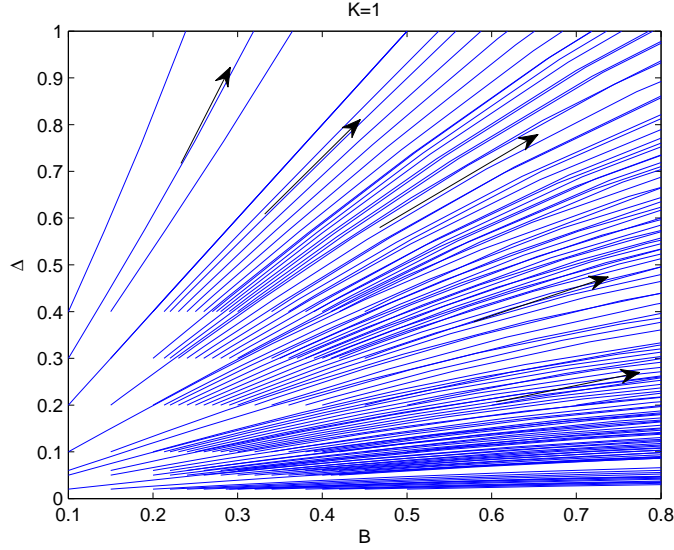


FIG. 2: (Colour online.) This figure shows the RG flow of  $\Delta$  with  $B$  for non-interacting helical liquid, i.e,  $K = 1$  in absence of umklapp scattering.

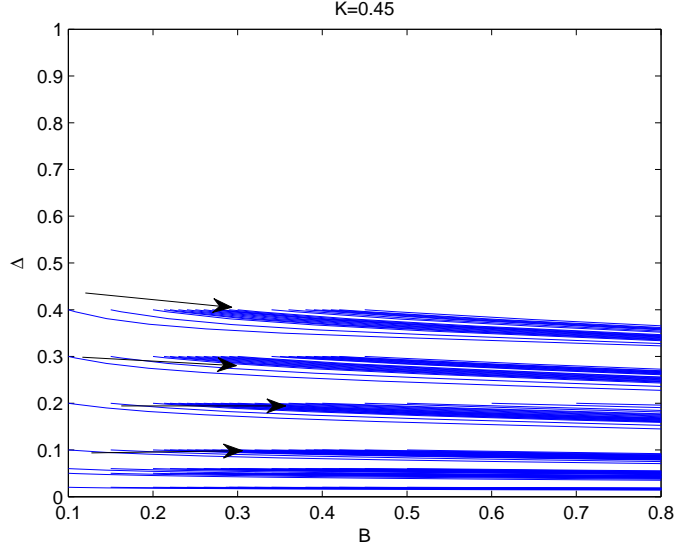


FIG. 3: (Colour online.) This figure shows the RG flow lines of  $\Delta$  with  $B$  for interacting helical liquid in absence of umklapp scattering. Here  $K = 0.45$  and  $B = 0.2$ .

coupling term,  $\Delta$ , flowing off to zero, i.e., for this case there is no existence of Majorana fermion mode in the system. The magnetic field induce ferromagnetic phase dictated by the direction of magnetic field is the only phase exist in the system.

Therefore, it is clear from the study of fig.2, fig.3 and fig.4 that the Majorana fermion mode

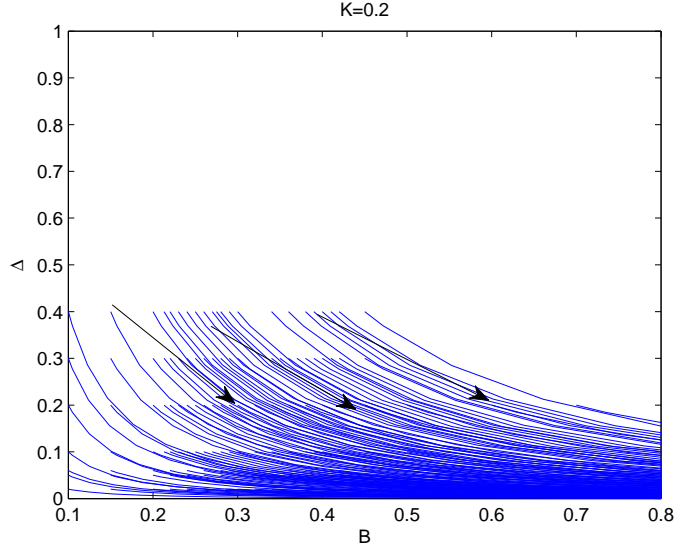


FIG. 4: (Color online.) This figure shows the RG flow lines of  $\Delta$  with  $B$  for strongly repulsive regime where  $K = 0.2$  in absence of umklapp scattering.

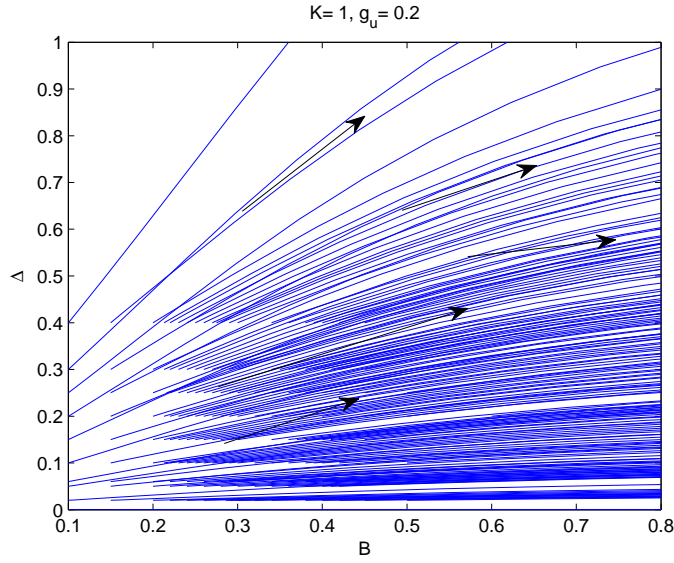


FIG. 5: (Colour online.) This figure shows the RG flow lines of  $\Delta$  with  $B$  for  $K = 1$  in presence of umklapp scattering,  $g_u = 0.2$ . Here  $B = 0.2$ .

only disappear in presence of very strong repulsion, otherwise, Majorana fermion mode is robust for weak and intermediate values of strong repulsion.

In fig.5 and fig.6, we present the results for  $g_u = 0.2$  and  $g_u = 0.4$  respectively for  $K = 1$ . It is very clear that umklapp scattering unable to destroy the superconducting phase,i.e.,

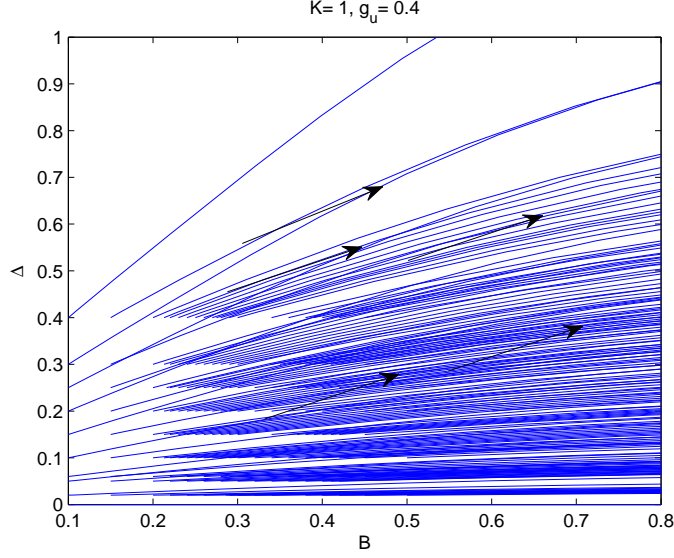


FIG. 6: (Colour online.) This figure shows the RG flow lines of  $\Delta$  with  $B$  for  $K = 1$  in presence of umklapp scattering,  $g_u = 0.4$ .

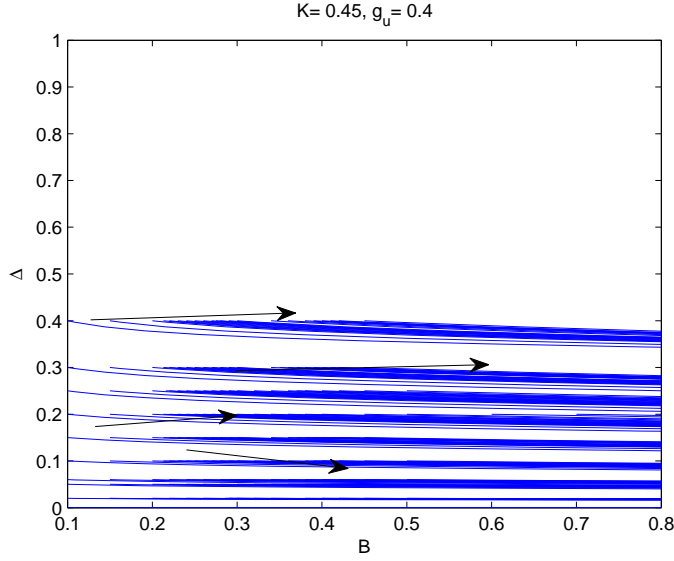


FIG. 7: (Colour online.) This figure shows the RG flow lines of  $\Delta$  with  $B$  for  $K = 0.45$  and  $g_u = 0.4$ .

the existence of Majorana phase is always in the system. The qualitative behaviour is the same for the both values of umklapp scattering.

In fig.7, we present the result of the variation of  $\Delta$  with  $B$  for  $K = 0.45$  and  $g_u = 0.4$ . It is clear to us that the behaviour of the system for repulsive regime is the same in presence

of umklapp scattering also. The magnitude of the proximity induce superconductivity gap decreases but it is nonzero, therefore the presence of Majorana fermion mode still exists. In the present situation, our system is topological insulator in the vicinity of an s-wave superconductor and an external magnetic field. Therefore the appearance of the topological phase, i.e., the appearance of Majorana fermion mode at the edge of the topological insulator depends on the proximity induce gap in the system. Here we study, the length scale dependent of the system and show how it appears. The survival condition for the Majorana fermion mode at the edge of the system is the following. If the superconducting region has a length  $L$ , the Majorana mode survive if the condition,  $L\Delta \gg 1$ , otherwise there is no Majorana fermion mode, i.e., the topological state of the system is absent [34]. Therefore, it is clear from the study fig.5, fig.6 and fig.7 that the presence of umklapp scattering does not affect the appearance of Majorana fermion mode.

In fig.8, we present the results of the study of  $\Delta$ ,  $B$  and  $K$  with the length scale ( $l$ ). Here we consider three values of  $\Delta = 0.1, 0.2, 0.4$  as the three initial values of  $\Delta$  but the initial value of  $B = 0.2$  and  $K = 1$ . It is clear from this length scale dependent study that the condition for the topological state of matter, i.e., the existence of Majorana fermion mode at the edge of the system.

In fig.9, we present the results of the study of  $\Delta$ ,  $B$  and  $K$  with the length scale under RG process. Here we consider  $K = 0.45$ , i.e, the repulsive regime of the system. It is clear from our study that  $B$  is increasing very rapidly with length scale. But for the higher values of  $\Delta$ s the length scale dependent is almost constant or decreases very slowly. It is clear from this figure that for the smaller values of  $\Delta$ s decreasing very rapidly with length scale and the condition for Majorana fermion mode existence is not satisfied for the large values of  $\Delta$ , the condition for Majorana fermion mode satisfies [34], and the system is in the topological state.

In. fig. 10, we present the variation for three different values of  $\Delta$ s with length under the RG process. The other parameters of the system are,  $K = 0.2$  and  $B = 0.2$ . For this parameter space, the system is in the strong repulsive phase and the proximity induce superconducting gap goes to zero value very rapidly. In this parameter space there is no evidence of Majorana fermion mode in the system.

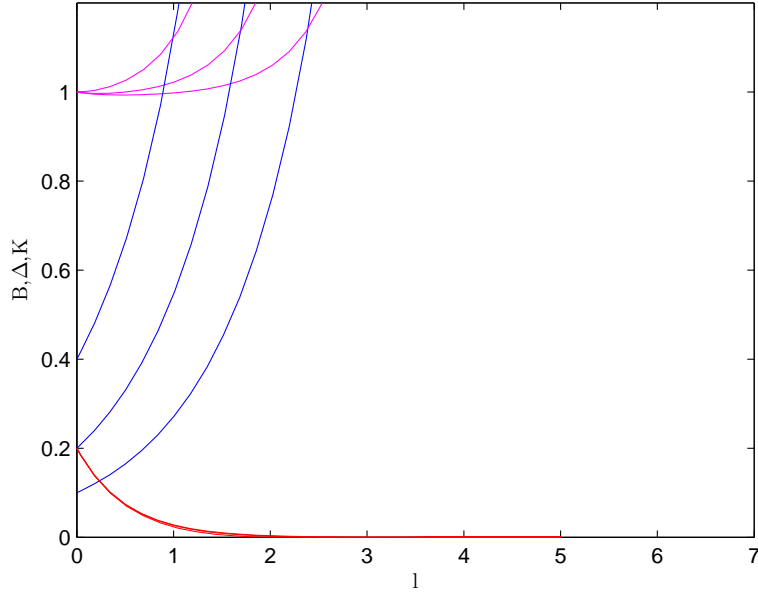


FIG. 8: (Colour online.) This figure shows the variation of  $\Delta$ ,  $B$  and  $K$  with length scale. The curves for blue, pink and red lines are respectively for the  $\Delta$ ,  $K$  and  $B$ .  $\Delta = 0.1, 0.2, 0.4$ ;  $B = 0.2$ ,  $K = 1$ ,

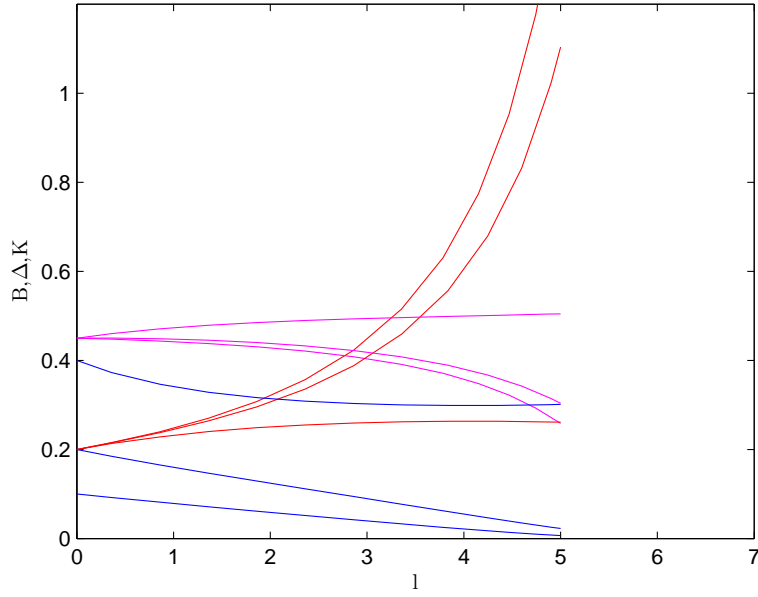


FIG. 9: (Colour online.) This figure shows the variation of  $\Delta$ ,  $B$  and  $K$  with length scale. The curves for blue, pink and red lines are respectively for the  $\Delta$ ,  $K$  and  $B$ .  $\Delta = 0.1, 0.2, 0.4$ ;  $B = 0.2$ ,  $K = 0.45$ ,

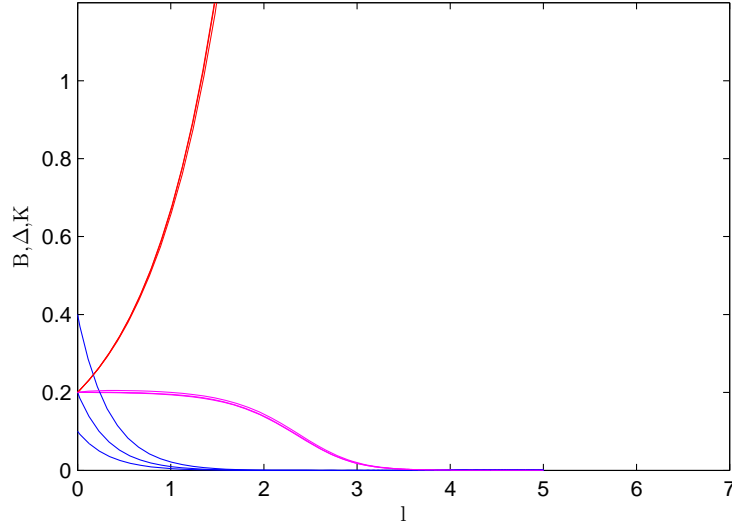


FIG. 10: (Colour online.) This figure shows the variation of  $\Delta, B$  and  $K$  with length scale. The curves for blue, pink and red lines are respectively for the  $\Delta, K$  and  $B$ .  $\Delta = 0.1, 0.2, 0.4$ ;  $B = 0.2, K = 0.2$ ,

### BEREZINSKII-KOSTERLITZ-THOULESS TRANSITION

Berezinskii, Kosterlitz and Thouless proposed that the disordering is facilitated by the condensation of topological defects [35, 36]. Generally one can consider it has as a vortex. It can be expressed as a two dimensional XY model. The orientation of a spin is define up to an interger multiple of  $2\pi$ . One can consider the spin configuration in which the travel of a closed path will see the angle rotate by  $2\pi n$  where  $n$  is the topological charge. Here we derive and discuss the appearance of Berezinskii-Kosterlitz-Thouless (BKT) transition in our system. BKT transition is a topological quantum phase transition in low dimensional many body system. Here we derive the quantum version of it, i.e., at  $T = 0$ .

To the best of our knowledge this is the first study to BKT transition in interacting helical liquid to find the physics of Majorana fermion mode.

Before we start to discuss the appearance of BKT transition in our system, we would like to discuss very briefly why it is necessary to study the BKT transition. Here we study two different situations of our model Hamiltonian. For the first case, the applied magnetic field is absent ( $B = 0$ ) and for the second case the proximity induced superconducting gap term is absent ( $\Delta = 0$ ). For both of these cases only one of the sine-Gordon coupling term is

present, therefore, there is no competition between the two mutually non local perturbation. Therefore one can think that there is no need to study the RG to extract the quantum phases and phase boundaries. But we still apply RG method for the following reason. Each of these Hamiltonians consist of two part, the first one ( $H_{01}$ ) is the non-interacting where the  $\phi$  and  $\theta$  fields show the quadratic fluctuations and the other part of these Hamiltonians are the sine-Gordon coupling terms which of either  $\theta$  or  $\phi$  fields. The sine-Gordon coupling term lock the field either  $\theta$  or  $\phi$  in the minima of the potential well. Therefore the system has a competition between the quadratic part of the Hamiltonian and the sine-Gordon coupling term and this competition will govern the low energy physics of these Hamiltonians in different limit of the system. The difference between the BKT transition and Majorana-Ising transition is that in BKT transition one transit from a gapped state to the gapless state (Luttinger liquid phase) but in Majorana-Ising quantum phase transition one transit from one gapped state to the other one.

For the non-interacting Helical liquid system, the system shows the two sets of RG equations for the different limit of the parameter space.

Here we derive the first set of BKT equations in absence of applied magnetic field ( $B = 0$ ).

$$\begin{aligned}\frac{d\Delta}{dl} &= (2 - \frac{1}{K})\Delta, \\ \frac{dK}{dl} &= -\Delta^2,\end{aligned}\tag{21}$$

To reduced this equations to the standard BKT equation, we do the following transformations.  $\tilde{K} = 1/2K$  and  $1 - \tilde{K} = -y_{||}$ . Then finally the above RG equations become,

$$\begin{aligned}\frac{d\Delta}{dl} &= -y_{||}\Delta, \\ \frac{dy_{||}}{dl} &= -\Delta^2,\end{aligned}\tag{22}$$

The other set of RG equations (when  $\Delta = 0$ ) are

$$\begin{aligned}\frac{dB}{dl} &= (2 - K)B \\ \frac{dK}{dl} &= -K^2B^2,\end{aligned}\tag{23}$$

Here we do the following two transformations,  $K = 2 + y_{||}$  and  $B \rightarrow B/2$ . Finally the equation reduce the standard BKT equation.



$$\begin{aligned}\frac{dB}{dl} &= -y_{||}B, \\ \frac{dy_{||}}{dl} &= -B^2,\end{aligned}\tag{24}$$

Any, BKT transition consists of three phase, the weak coupling phase, strong coupling phase and the intermediate coupling where the system transits from the weak coupling phase to strong coupling phase. The structure of these RG equations have the general structure for the different sets of RG equations.

The analysis of these RG equations consists of three different phases.

The first one is the weak coupling phase when  $y_{||} > \Delta$ . In this phase there is no gapped excitation of the system, i.e, the system is in the Luttinger liquid phase. In this phase, there is no evidence of Majorana fermion mode, i.e., the system is in the non-topological state.

The second one is the crossover regime, the mathematical condition for this crossover regime is  $-\Delta < y_{||} < \Delta$  ( $-\Delta < (\frac{1}{2K} - 1) < \Delta$ ). One observe the crossover from the weak coupling phase ( $y_{||} = |\Delta|$ ) to the strong coupling regime ( $y_{||} = -|\Delta|$ ). During this phase cross over system transit from Luttinger liquid phase to the proximity induced superconducting gap phase, i.e.,  $\Delta \neq 0$ . In this phase the system has the Majorana fermion mode. For this situation, system transit from non-topological state to the topological state.

The third one is the deep massive phase, the RG flows flowing off to the strong coupling regime away from the Gaussian fixed line.

The structure of the second BKT equations (Eq.23 ) is the same form as that of the first one (Eq. 21). Therefore the analysis of these equation are the same as that of the first one. The only difference is that here is no evidence Majorana fermion mode in this system. The system only shows the ferromagnetic phase.

The physics of BKT transition has studied in different condensed matter system like in one or two dimensional superconductivity, superfluidity, Josephson junction array, melting of cryastaline thin flim, one-dimensional metal and quantum magnets. But the explicit study of BKT transition for cavity QED lattice is absent in the previous literature.

## EFFECT OF CHEMICAL POTENTIAL

Now we consider the effect of the presence of chemical potential ( $\mu$ ) in the physics of interacting helical liquid. At first we consider the absence of umklapp scattering ( $g_u$ ) term and the proximity induce superconducting gap ( $\Delta$ ). At first we consider a transformation,  $2\phi \rightarrow 2\phi + \delta_1 x$ , where  $\delta_1 = -\frac{K\mu}{\pi}$ . This transformation eliminate the term  $\partial_x \phi(x)$  from the bosonized Hamiltonian but this transformation leads to a spatially oscillating term, i.e.,  $\frac{B}{\pi} \cos(\sqrt{\pi}(2\phi + \delta_1 x)) = \frac{B}{\pi} \cos(2\sqrt{\pi}\phi + \delta_1 \sqrt{\pi}\phi(x))$ . In the absence of  $g_u$  and  $\Delta$ , system shows the commensurate to incommensurate transition.

If  $\delta_1 a \gg 1$ , then the term  $\frac{B}{\pi} \cos(\sqrt{\pi}(2\phi + \delta_1 x))$  is quickly oscillating and average out to zero which reflect the competition between the  $\mu$  and  $B$ . As a result of this competition, the RG flow in  $B$  has to be cut-off when  $2\delta_1(l) \sim 1$ . To the lowest order in  $B, \Delta$  and  $\delta_1$ , the RG flows of  $\delta_1$  is

$$\frac{d\delta_1}{dl} = \delta_1. \quad (25)$$

When the all perturbations are relevant, they flows to the strong coupling phase under RG transformation. If the coupling  $B(l)$  reaches the strong coupling phase before  $\delta_1(l)a$  reach to one. The phase of the system is the ferromagnetic phase. This condition is  $2\delta_1(l^*)a \ll 1$  which translate into the requirement  $\delta_1(0)a \ll B(0)^{1/(2-K)}$ . When we consider the opposite scenario, i.e., the  $B$  term is not able to reach the strong coupling phase but the  $\Delta(l)$  term reaches the strong coupling phase and system shows the presence of Majorana fermion mode. The presence of  $\mu$  has no effect on it, because the  $\mu$  term is related with the  $\phi$  field and the proximity induce superconductivity is related with  $\theta$  term. Now we discuss the physics in presence of umklapp term, the system posses a oscillatory term  $\frac{g_u}{\pi} \cos(4\sqrt{\pi}\phi(x) + 2\delta_1 \sqrt{\pi}\phi(x))$ . If the  $g_u(l)$  term reaches the strong coupling phase earlier than the  $B(l)$  then the system is in the ferromagnetic phase.

## SUMMARY AND CONCLUSIONS

We have done the renormalization group study of interacting helical liquid in presence of proximity induce superconductivity and applied magnetic field along the edge direction. We have found that Majorana fermion mode exist for this system even in the presence

of repulsive interaction and umklapp scattering. We have also studied the length scale dependence of  $B, \Delta, K$  under the renormalization to get the condition for the existence of Majorana fermion explicitly. We have also studied the Berezinskii-Kosterlitz-Thouless transition. The condition of self duality destroy in presence of umklapp scattering.

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## Appendix

The authors of Ref. [12] have not presented the detail derivation of the bosonized Hamiltonian. Here we present the detail derivation of the bosonized Hamiltonian during this derivation we follow the following references [33, 37].

In the bosonization process, one can express the fermionic field of low dimensional quantum many body system as,

$\psi_{R/L,\uparrow} = \frac{1}{2\pi\alpha}\eta_{R,\uparrow} e^{i\sqrt{4\pi}\phi_{R,\uparrow/\downarrow}}$ .  $\eta_{L/R}$  is the Klein factor to preserve the anticommutivity of the fermionic field which obey the Clifford algebra (here cite reference). Here we introduce the two bosonic fields,  $\theta$  and  $\phi$ , which are dual to each other. These two fields are related with the following relations,  $\phi = \phi_R + \phi_L$  and  $\theta = \theta_R + \theta_L$ .

The analytical relation between the Klein factors have mentioned in Ref. [33, 37].

The non-interacting Hamiltonian for helical liquid is

$$\begin{aligned} H_0 &= \psi_{L,\downarrow}^\dagger (iv_F \partial_x - \mu) \psi_{L,\downarrow} + \psi_{R,\uparrow}^\dagger (-iv_F \partial_x - \mu) \psi_{R,\uparrow} \\ &= v_F [(\partial_x \phi_{L,\downarrow})^2 + (\partial_x \phi_{R,\uparrow})^2] - \frac{\mu}{\sqrt{\pi}} \partial_x \phi. \end{aligned} \quad (26)$$

We use the following relation during the derivation of the above Hamiltonian.

$$\rho_{R/L,\sigma} = \frac{1}{\sqrt{\pi}} \partial_x \phi_{R/L,\sigma}.$$

Now we bosonize the forward scattering and umklapp scattering term, these two interaction terms are time reversal symmetry invariant terms.

$$\begin{aligned} H_f &= g_2 \psi_{L,\downarrow}^\dagger \psi_{L,\downarrow} \psi_{R,\uparrow}^\dagger \psi_{R,\uparrow} + \frac{g_4}{2} [(\psi_{L,\downarrow}^\dagger \psi_{L,\downarrow})^2 + (\psi_{R,\uparrow}^\dagger \psi_{R,\uparrow})^2] \\ &= g_2 \rho_{L\downarrow} \rho_{R\uparrow} + \frac{g_4}{2} [(\rho_{L\downarrow})^2 + (\rho_{R\downarrow})^2] \end{aligned} \quad (27)$$

Now we use the bosonized version of the density operator to express the bosonized form of the above Hamiltonian. Finally the above Hamiltonian reduced to

$$H_f = \frac{g_2}{\pi} \partial_x \phi_{L\downarrow} \partial_x \phi_{R\uparrow} + \frac{g_4}{2\pi} [(\partial_x \phi_{L\downarrow})^2 + (\partial_x \phi_{R\uparrow})^2] \quad (28)$$

Now we consider the umklapp term, we follow the convention of Wu, Bernevig and Zhang [25], to write the umklapp term in a point splitted form to regularize the theory where lattice constant use as a ultra-violet cut-off [25].

$$\begin{aligned}
H_{um} &= -g_u[(\psi_{L\downarrow}^\dagger \psi_{R\uparrow})^2 + (\psi_{R\uparrow}^\dagger \psi_{L\downarrow})^2] \\
&= -\frac{-g_u}{(2\pi)^2}[\eta_{L\downarrow}\eta_{R\uparrow}\eta_{L\downarrow}\eta_{R\uparrow}e^{2i\sqrt{4\pi}(\phi_{L\downarrow}+\phi_{R\uparrow})}] + \frac{-g_u}{(2\pi)^2}[\eta_{R\uparrow}\eta_{L\downarrow}\eta_{R\uparrow}\eta_{L\downarrow}e^{-2i\sqrt{4\pi}(\phi_{L\downarrow}+\phi_{R\uparrow})}] \quad (29)
\end{aligned}$$

Now we derive the bosonized form of the applied magnetic field along the edge direction.

$$\begin{aligned}
H_B &= \frac{B}{2\pi}(\eta_{L,\downarrow}\eta_{R,\uparrow}e^{i\sqrt{4\pi}(\phi_{L\downarrow}+\phi_{R\uparrow})+h.c}) \\
&= \frac{B}{2\pi}\eta_{L,\downarrow}\eta_{R,\uparrow}(e^{i\sqrt{4\pi}\phi} - e^{-i\sqrt{4\pi}\phi}) \quad (30)
\end{aligned}$$

$$\begin{aligned}
H_\Delta &= \frac{\Delta}{2\pi}(\eta_{L,\downarrow}\eta_{R,\uparrow}e^{i\sqrt{4\pi}(-\phi_{L\downarrow}+\phi_{R\uparrow})+h.c}) \\
&= \frac{\Delta}{2\pi}\eta_{L,\downarrow}\eta_{R,\uparrow}(e^{i\sqrt{4\pi}\theta} - e^{-i\sqrt{4\pi}\theta}) \quad (31)
\end{aligned}$$

Now we follow the prescription of Senechal to choose a Klein basis which simultaneously diagonalizes all Klein factors. Here we are facing two Klein factors

$\kappa_1 = \eta_{L\downarrow}\eta_{R\uparrow}\eta_{L\downarrow}\eta_{R\uparrow}$ .  $\kappa_2 = \eta_{L\downarrow}\eta_{R\uparrow}$ . According to the notation of Senechal, we can write,  $\eta_{L\downarrow} = 1 \otimes \sigma^y$ ,  $\eta_{R\uparrow} = \sigma^x \otimes \sigma^x$ , Therefore one can write the  $\kappa_1$  and  $\kappa_2$  as

$$\kappa_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \kappa_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}.$$

The above two matrices are commute with each other, therefore one can diagonalize these matrices simultaneously. The eigen vectors of one matrix can be expressed as a linear combination of eigen vectors of other matrix from which one can construct the unitary matrix which diagonalize the both matrix  $\kappa_1$  and  $\kappa_2$  simultaneously. The diagonalize matrix, we may call it as a  $\kappa_3$  and  $\kappa_4$ .

$$\kappa_3 = U^{-1}\kappa_1U = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \kappa_4 = U^{-1}\kappa_2U = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix},$$

One can gauge choice by considering,  $-1$  and  $-i$  from the upper left corner and forget the rest of the Klein space.

Using this convention one can write the Hamiltonian  $H_B$  (Eq. 30) and  $H_\Delta$  (Eq.31) are

respectively.

$$H_B = \frac{B}{\pi} \sin(\sqrt{4\pi}\phi(x)).$$

$$H_\Delta = \frac{\Delta}{\pi} \sin(\sqrt{4\pi}\theta(x)).$$

$$H_{g_u} = \frac{g_u}{2\pi^2} \cos(\sqrt{16\pi}\phi(x)).$$

$$v = v_F + \frac{g_4}{2\pi}, \quad K = 1 - \frac{g_2}{2\pi v_F}.$$

Explanation of signe mismatch:

The author of Ref. [12] have considered  $\sqrt{4\pi}\phi \rightarrow \sqrt{4\pi}\phi + \pi/2$ ,  $\sqrt{4\pi}\theta \rightarrow \sqrt{4\pi}\theta - \pi/2$ .

Therefore the form of the full bosonized Hamiltonian is

$$\begin{aligned} H = & \frac{v}{2} \left( \frac{1}{K} (\partial_x \phi)^2 + K (\partial_x \theta)^2 \right) - \frac{\mu}{\pi} \partial_x \phi \\ & + \frac{B}{\pi} \cos(\sqrt{4\pi}\phi) - \frac{\Delta}{\pi} \cos(\sqrt{4\pi}\theta) + \frac{g_u}{2\pi^2} \cos(4\sqrt{\pi}\phi) \end{aligned} \quad (32)$$